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CONFIGURATIONS OF TWO D-INSTANTONS.

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ABSTRACT

The potential between two separated D-instantons at fixed (super) space-time points is obtained by a simple explicit integration over the 'massive' variables of the zero-dimensional reduction of ten-dimensional $U(2)$ super Yang–Mills theory. This potential vanishes for asymptotically large separations, becoming significant at separations of around the ten-dimensional Planck scale with a singularity at the origin, which is resolved by the extra 'massless' internal Yang–Mills super-coordinates.

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The emerging unified description of superstring theory and eleven-dimensional supergravity is largely based on appreciation of the p-brane solitonic solutions of these theories and their compactification to lower dimensions. The D-brane description of stringy p-branes that carry $\mathrm{R} \otimes \mathrm{R}$ charges [1] has given a well-controlled perturbative formulation of this class of stringy soliton and has lead to a magnificent set of insights into stringy effects, black hole quantum mechanics and properties of supersymmetric quantum field theory. Compactification of euclidean p-brane world-volumes leads to instanton configurations that also play significant rôles in the complete theory (as explored in, for example, [2, 3, 4, 5].

In addition to the solitonic branes, type IIB superstring theory possesses an instanton solution in ten dimensions – the $p = -1$ D-brane that couples to the pseudoscalar $\mathrm{R} \otimes \mathrm{R}$ charge. Such instantons presumably have important effects in the theory – for example, they are responsible for 'point-like' effects in fixed-angle scattering. Indeed, the possibility of inducing point-like structure was one of the original reasons for studying string theories with Dirichlet boundary conditions in both the bosonic theory [6, 7] and the superstring [8]. In a separate paper [9] we discuss such point-like scattering in the background of a D-instanton and also determine certain non-perturbative terms that are induced in the effective potential by the D-instanton background.

Here we will focus on multi D-instanton configurations in uncompactified ten-dimensional space-time. The classical D-instanton solutions of IIB supergravity theory in [10] have the interpretation, in the string frame, of Einstein–Rosen space-time wormholes. Some integer unit of $\mathrm{R} \otimes \mathrm{R}$ pseudoscalar flux disappears through any given wormhole into an unphysical universe. Thus, the sector of the moduli space with charge N has $p(N)$ wormholes ending on distinct unphysical universes, where $p(N)$ is the partition of N into integers. The classical description is obviously inadequate since the string coupling gets large in the wormhole neck at around the Planck scale. To go further we need to analyze the stringy description of the D-instanton configuration space.

A configuration of separated D-instantons is described by world-sheets with boundaries fixed at space-time points representing the location of the instantons [11, 12]. Such world-sheets can be described by gluing open-string strips together, where the open 'strings' have end-points fixed in *space-time*. This is conformally equivalent to the insertion of point-like densities on closed strings [6]. The sum over world-sheets with fixed positions for the instantons is equivalent to integrating over the 'stretched open strings' joining them. However, when D-instantons coincide there are enhanced symmetries and these integrations are singular.

The moduli space of n D-branes with world-volumes of dimension $p+1$ is described by the reduction to $p+1$ dimensions of ten-dimensional supersymmetric $U(n)$ Yang–Mills theory [14]. In other words the Yang–Mills potential, $A_{A\mu}$, and the Majorana–Weyl fermion, ψ_A, are taken to be independent of the $9-p$ dimensions transverse to the world-

volume.[3] The D-instanton configuration space is determined by the $p = -1$ case. This is a model of $n \times n$ bosonic and fermionic $U(n)$ matrices, $A_\mu = A_{A\mu} T^A$ and $\psi = \psi_A T^A$, where T^A are the generators of the Lie algebra of $SU(N)$ satisfying $[T^A, T^B] = f^{ABC} T^C$ with f^{ABC} being the structure constants. In addition there is a $U(1)$ which describes the overall centre of mass degrees of freedom – since all variables are in the adjoint of the group, they are all uncharged under this $U(1)$. The D-instantons are solutions of the euclidean theory so the Minkowski signature fields should be Wick rotated in order to give a well-defined measure. In the latter part of the paper we will adopt the procedure of carrying out as much of the calculation as possible with Minkowski signature in order to make use of well-known manipulations of $SO(9,1)$ Majorana–Weyl fermions and Dirac Γ matrices.

The D-instanton action is given by simply deleting all derivatives in the supersymmetric Yang–Mills action,

$$S = \frac{1}{2\kappa} \left(\frac{1}{4} \mathrm{tr}([A_\mu, A_\nu]^2) + \frac{i}{2} \mathrm{tr}(\bar{\psi} \Gamma^\mu [A_\mu, \psi]) \right), \tag{1}$$

where κ is the string coupling constant. The minimum potential of the system is given by setting the fermions to zero and $[A_\mu, A_\nu] = 0$. These conditions are satisfied by matrices A_μ in the $U(1)^n$ Cartan subalgebra which have the form,

$$A_\mu = \begin{pmatrix} Y_1^\mu & & & \\ & Y_2^\mu & & \\ & & .. & \\ & & & Y_n^\mu \end{pmatrix}, \tag{2}$$

where $I = 1, \ldots, n$ and the Y_r^μ are the positions of n separated D-instantons. The partition function is simply an integral of e^{-S} over bosonic and fermionic matrices. The measure on the reduced moduli space of n separated instantons is obtained by integrating all variables that are not in the Cartan subalgebra, so that the partition function can be expressed as,

$$\mathcal{Z}^{(n)} = \int \prod_{I=1}^{n} d\psi_I dA_I Z^{(n)}[\psi_I, A_I] e^{J^I \psi_I}, \tag{3}$$

where J^I represent closed-string sources and

$$Z^{(n)}[\psi_I, A_I] = \int \prod_{A'=1}^{n^2-n} d\psi_{A'} dA_{A'} \ \exp\left(-S(A, \psi)\right), \tag{4}$$

and $_{A'}$ indicates that the elements in the Cartan subalgebra are not included.

[3] Our conventions are that the index A denotes the adjoint of $U(n)$ and takes n^2 values, $\mu = 0, 1, \cdots, 9$ is a $SO(9,1)$ vector index and $a = 1, \cdots, 16$ is a $SO(9,1)$ Weyl spinor index.

The measure $Z^{(n)}$ is trivially independent of the variables in the overall diagonal $U(1)$ so that the partition function (3) includes a volume factor due to integration over the corresponding ten bosonic and sixteen fermionic variables (which are identified with the sixteen broken supersymmetries). We are here thinking of soaking up the overall fermionic variables with external sources which enter in the calculation of closed-string scattering amplitudes [9]. We want to evaluate the measure for the remaining integrations in (3), which is given by the integration over the 'internal variables' in (4). This corresponds to the situation in which the D-instantons are all separated and we are integrating over the 'stretched strings' joining them.

We will now specialize to the case of two D-instantons, for which the relative moduli space is determined by the group $SU(2)$. In order to simplify the expression it will later prove very convenient to make use of symmetries of the action (1) in order to pick a particular parameterization of the integration variables. One of these symmetries is the remnant of the local $SU(2)$ gauge symmetry, with parameter Λ^A ($A = 1, 2, 3$). In addition, the action is invariant under $SO(9, 1)$ Lorentz transformations with parameter $\omega_{\mu\nu}$ and supersymmetry transformations with a sixteen-component Majorana-Weyl spinor Grassmann parameter, ϵ. These supersymmetries are those that are unbroken by the presence of a single D-instanton. The transformations of the variables that leave the action invariant are

$$\begin{aligned} \delta A_{A\mu} &= \epsilon^{ABC} A_{B\mu} \Lambda^C + \omega_{\rho\lambda} (\Sigma^{\rho\lambda})^{\nu}{}_{\mu} A_{A\nu} + i\bar{\epsilon}\Gamma^{\mu}\psi_A && (5) \\ \delta\psi_A &= \epsilon^{ABC}\psi_B\Lambda^C + \omega_{\rho\lambda}\Gamma^{\rho\lambda}\psi_A + \epsilon^{ABC} A_{B\mu} A_{C\nu} \Gamma^{\mu\nu}\epsilon. && (6) \end{aligned}$$

The vector $A_{3\mu}$ will be taken as the element of the $U(1)$ Cartan subalgebra which is associated with the relative position of the two instantons. This choice breaks the $SU(2)$ symmetry.

We will see that there are two distinct types of terms that contribute to the partition function so that ,

$$\mathcal{Z}^{(2)} = \mathcal{Z}_1^{(2)} + \mathcal{Z}_2^{(2)}. \tag{7}$$

The first term arises by soaking up all sixteen components of ψ_3 with external sources and has the form,

$$\mathcal{Z}_1^{(2)} = \int d^{10} A_3 (J^3)^{16} e^{-V(|A_3|)} \tag{8}$$

The second term on the right-hand side of (9) is proportional to $(J^3)^8$ and arises because the action, S, is independent of half the ψ_3's. Since there is no covariant way of eliminating half a spinor we will make use of a light-cone Minkowski space description which gives a result of the form,

$$\mathcal{Z}_2^{(2)} = \int d^{10} A_3 (J^3)^8 X(A_3) e^{-V(|A_3|)}, \tag{9}$$

where $X(A_3)$ is a function to be determined later.

We will first calculate the function $\exp(-V(|A_3|)$, which involves setting $\psi_3 = 0$ in the action (1). For the moment we will treat the problem in arbitrary dimension d. For fixed A_3 the action is simplified by choosing coordinate axes so that $A_3^d = L$ and $A_3^i = 0$ for $i = 1, \cdots, d-1$, so that

$$\int d^d A_{3\mu} = \int dL L^{d-1} d\Omega^{d-1}, \tag{10}$$

where $d\Omega^{d-1}$ is the volume of the unit $(d-1)$-sphere. The dimensionality of the spinor will be denoted by p, where $p = 2^{(d-2)/2}$ for minimal spinors in $2, 3, 4, 6$ or 10 dimensions. The integrations over the internal fermions are very simple, giving a factor of

$$\int d^p\psi_1 d^p\psi_2 \exp(\kappa^{-1} L\bar{\psi}_2 \Gamma^d \psi_1) = \kappa^{-p} L^p. \tag{11}$$

Now we can integrate out the internal bosonic variables $A_{1\mu}$ and $A_{2\mu}$. Recall that in this discussion these are $SO(d)$ vectors. The bosonic action reduced to zero dimensions is given by,

$$S = \frac{1}{4\kappa} \left\{ L^2 (A_1)^2 + L^2 (A_2)^2 + (A_1)^2 (A_2)^2 - (A_1 \cdot A_2)^2 + (a_1 A_2 - a_2 A_1)^2 \right\}, \tag{12}$$

where A_1^i and A_2^i are $(d-1)$-vectors and $a_1 = A_1^d$ and $a_2 = A_2^d$. The $SO(d-1)$ symmetry of this action can be used to write the integration measure for the internal integrations as,

$$\int d^d A_1 d^d A_2 = \int da_1 da_2 \int dR_1 dR_2 d\Omega_1^{d-2} d\Omega_2^{d-3} d\theta R_1^{d-2} R_2^{d-2} (\sin\theta)^{d-3}, \tag{13}$$

where $|A_1| = R_1$, $|A_2| = R_2$ and $|A_1 \cdot A_2| = R_1 R_2 \cos\theta$. The action (12) expressed in these variables is,

$$S = \frac{1}{4\kappa} \left\{ L^2 \left(R_1^2 + R_2^2 \right) + R_1^2 R_2^2 (\sin\theta)^2 + a_1^2 R_2^2 + a_2^2 R_1^2 - 2a_1 a_2 R_1 R_2 \cos\theta \right\}. \tag{14}$$

The gaussian integration over a_1 and a_2 is simple to evaluate, leading to a factor of $\kappa\pi(R_1 R_2 \sin\theta)^{-1}$. Defining $x = (4\kappa)^{-1} L^2 R_1^2$ and $y = (4\kappa)^{-1} L^2 R_2^2$, the partition function becomes,

$$\begin{aligned} e^{-V(L)} &= \kappa^{d-1-p} L^{4-2d+p} \int dx dy d\theta (xy \sin^2\theta)^{(d-4)/2} e^{\left(-(x+y) - 4\kappa L^{-4} xy \sin^2\theta\right)} \\ &= \kappa^{d-1-p} L^{4-2d+p} \int dy d\theta \frac{y^{(d-4)/2} (\sin\theta)^{d-4}}{(1 + 4\kappa L^{-4} y \sin^2\theta)^{(d-2)/2}} e^{-L^2 y}. \end{aligned} \tag{15}$$

Here, and in subsequent equations, we ignore an overall constant factor in $\mathcal{Z}^{(2)}$ which is proportional to the volume of the bosonic integrations and is independent of κ and L.

Using (for even d)

$$\int_0^\pi d\theta \frac{(\sin\theta)^{(d-4)}}{(a^2+b^2\sin^2\theta)^{(d-2)/2}} = \frac{(d-3)!!}{2^{(d-2)/2}\,((d-2)/2))!}\frac{1}{a(b^2+a^2)^{(d-3)/2}} \tag{16}$$

(see, for example, [15] section 3.642 eq.(3)), it follows that the potential $V(L)$ is given by,

$$e^{-V(L)} = \kappa^{(2d-3p)/4}\hat{L}^{4-2d+p}\int_0^\infty dy \frac{y^{(d-4)/2}}{(1+4y\hat{L}^{-4})^{(d-3)/2}}e^{-y}, \tag{17}$$

where $\hat{L} = \kappa^{-1/4}L$.

In the limit of large separations, $L \to \infty$, the potential in (17) behaves as,

$$V(L) \to (2d-4-p)\ln\hat{L} + O(\hat{L}^{-4}). \tag{18}$$

For pure Yang–Mills theory ($p = 0$) in $d > 2$ this is badly behaved (and $\mathcal{Z}_1^{(2)}$ is not extensive) but for the supersymmetric Yang–Mills theories, which have $d = 3, 4, 6$ or 10 and $p = 2, 4, 8$ and 16 respectively, the potential vanishes asymptotically and the partition function is proportional to the volume, $\int d^dA_3$. Furthermore, in the nonsupersymmetric cases a non-zero metric is generated for the fluctuations $\delta A_{3\mu}\delta A_{3\nu}$, which arises from the effect of integrating over the internal bosons. In the supersymmetric cases the internal fermion integrations cancel this and the metric remains flat.

The exact expression for the potential shows a changeover due to higher order κ effects at distances of order $\hat{L} \sim 1$, or $L \sim \kappa^{1/4}$. Recalling that the ten-dimensional Planck distance is given, in string theory, by

$$l_P = \kappa^{1/4}\sqrt{\alpha'} \tag{19}$$

(the string scale $\sqrt{\alpha'}$ has been set equal to one in the forgoing discussion) we see that the two-instanton potential develops non-trivial dependence at around the Planck scale, which is the scale at which the string coupling becomes strong in the classical D-instanton solution. Precisely how this is reflected in physical processes is not apparent from this analysis but it seems likely to be of importance in understanding the dynamics of string theory more completely.

The potential, $V(L)$, is a free energy that encodes the sum over all loop diagrams, which can be recovered by expanding it in a power series in κ/L^4. This can be seen to reproduce the sum over 'Feynman diagrams' derived from the 'action', (1), when ψ^3 is soaked up by external sources. One simple way of determining the coefficients in this series is to express (17) in terms of the parabolic cylinder function, $D_{3-d}(w)$ (defined in [15] section 3.383 eq.7),

$$e^{-V(L)} = (const.)\kappa^{(2d-3p)/4}w^{(p-2)/2}e^{w^2/4}D_{3-d}(w), \tag{20}$$

where $w^2 = \hat{L}^4/2$. The second-order differential equation satisfied by $D_n(w)$ ([15] 9.255 eq.3) leads to,

$$\frac{d^2V}{dw^2} - \left(\frac{dV}{dw}\right)^2 - \left(w + (p-2)\frac{1}{w}\right)\frac{dV}{dw} - \frac{p(p-2)}{4w^2} - \frac{p+4-2d}{2} = 0. \tag{21}$$

This equation can be solved iteratively in powers of w^{-2}, giving a series that summarizes the sum over all closed-string world-sheets with boundaries that are fixed (in all directions) on either of the two D-instantons. As in the case of the bosonic string [12], the logarithm of the partition function is a sum of *connected* world-sheets. Importantly, it is only for the supersymmetric cases that the constant term is absent from (21). In that case $V = 0$ as $w \to \infty$, which corresponds to the vanishing of the lowest-order diagram – the cylinder diagram which vanishes by the abstruse Jacobi identity due to the cancellation of the loop of internal bosons and the internal fermions [8]. This is the same as the vanishing of the one-loop diagram in the super-Yang–Mills theory. In the non-supersymmetric case the constant term in (21) leads to a logarithmic divergence in the potential at large L – correspondingly, the cylinder diagram is non-zero in the bosonic string theory. Each successive term adds a boundary in the sum over world-sheets (which inserts a vertex and two propagators in the Feynman diagrams for the Yang–Mills theory). These higher-order diagrams are non-vanishing.

In the classical field theoretic description of [10] the configuration of two singly charged wormholes does not merge smoothly into a single doubly charged wormhole when the relative separation vanishes. However, this is the region in which the classical solution is not expected to be a good guide. From the integral (17) it is clear that near the origin, $L = 0$, the potential behaves as

$$V(L) \sim -(p-2)\ln \hat{L}. \tag{22}$$

This singularity is the signal that the variables A_3 and ψ_3 are not a complete description of the moduli space. At the origin the 'stretched strings' become important as their 'mass' vanishes – in particular, the action becomes independent of the fermionic variables ψ_1 and ψ_2 when $L = 0$ so that they are supermoduli, and integrating over them causes $Z^{(2)}$ to vanish. This resolution of the apparent singularity at points in moduli space where instantons coincide is characteristic of the D-brane picture.

The preceding discussion applies to the $(J^3)^{16}$ term in which all sixteen components of the spinor, ψ_3, are soaked up (from now on we will stay with the ten-dimensional case). However, supersymmetry implies that the action depends on eight components of ψ_3 so that half of the components can be integrated in (3) and there are only eight independent supermoduli, which is half the number of components in a ten-dimensional Weyl spinor. This gives the $\mathcal{Z}_2^{(2)}$ term in (7). To exhibit this explicitly requires a non-covariant choice of coordinates which will be motivated here by the light-cone description of D-instantons in

string theory [8, 13]. There should presumably be a more covariant method of describing the supermoduli that involves extra gauge degrees of freedom, such as those that enter in the κ-symmetry of the manifestly covariant formulation of superstring theory.

We will begin this part of the discussion with lorentzian signature and define light-cone coordinates with respect to two particular directions so that a $SO(9,1)$ vector decomposes into $SO(8)\times U(1)$,

$$\begin{array}{ccl} 10 & \to & 8_0 \oplus\ 1_1 \oplus\ 1_{-1} \\ A_\mu & \to & A^i,\quad A^+,\quad A^- \end{array} \tag{23}$$

where $A^\pm = (A^0 \pm A_9)/\sqrt{2}$. The inner product of two vectors is $A_\mu B^\mu = A_+B^+ + A_-B^+ + A_iB_i$ $(i=1,\cdots,8)$ where $A_+ = -A^-$ and $A_- = -A^+$. The $SO(9,1)$ gamma matrices, Γ^μ, satisfy $(\Gamma^+)^2 = (\Gamma^-)^2 = 0$ and the expressions $\Gamma^+\Gamma^-/2$ and $\Gamma^-\Gamma^+/2$ are projectors that decompose a sixteen-component chiral $SO(9,1)$ spinor into the two inequivalent $SO(8)$ spinors, $\mathbf{8_s}$ and $\mathbf{8_c}$. The two inequivalent $SO(8)$ spinors which will be represented by undotted and dotted eight-component spinor indices so that the supersymmetry moduli decompose as follows,

$$\begin{array}{ccl} 10 & \to & 8_{-\frac{1}{2}} \oplus 8_{\frac{1}{2}} \\ \psi & \to & \psi^a,\qquad \dot\psi^{\dot a} \end{array} \tag{24}$$

where $a, \dot a = 1,\ldots,8$. From now on all spinors will be dotted or undotted $SO(8)$ spinors and the gamma matrices, Γ^μ, will be decomposed into $SO(8)$ matrices, $\gamma^i_{a\dot b}$ and $\gamma^i_{\dot a b}$ (and the index a will from now on take eight values).

We will use the $SU(2)$ symmetry, $\delta A^+_a = \epsilon^{abc}A^+_b\Lambda^c$, to rotate to the 'light-cone gauge',

$$A^+_1 = 0 \quad , \quad A^+_2 = 0. \tag{25}$$

This means that the integration variables A^+_1 and A^+_2 are replaced by Λ^2 and Λ^1, with a jacobian factor of

$$\left|\frac{\delta A^+_1}{\delta\Lambda^2}\frac{\delta A^+_2}{\delta\Lambda^1}\right| = (A^+_3)^2. \tag{26}$$

We now wish to transform away other variables by making use of those $SO(9,1)$ and supersymmetry transformations which do not affect the condition (25). We first consider the supersymmetry transformations,

$$\delta\dot\psi_3 = A^i_1A^j_2\gamma^{ij}\dot\rho + (A^-_1A^i_2 - A^-_2A^i_1)\gamma^i\rho, \tag{27}$$

$$\delta\psi_3 = A^i_1A^j_2\gamma^{ij}\rho. \tag{28}$$

The light-cone gauge conditions (25) are spoilt by the transformations associated with the ρ components but the $\dot\rho$ components can be used to eliminate the dotted components by setting

$$\dot\psi^{\dot a}_3 = 0. \tag{29}$$

This replaces the $\psi_3^{\dot{a}}$ integrations with integrations over the components of $\dot{\rho}^{\dot{a}}$ with a jacobian,

$$\left|\frac{\partial \dot{\psi}^{\dot{a}}}{\partial \dot{\rho}^{\dot{b}}}\right| = \frac{1}{Pf(A_1^i A_2^j \gamma_{\dot{a}\dot{b}}^{ij})}, \tag{30}$$

where Pf denotes the Pfaffian, which can easily be evaluated explicitly to give,

$$Pf(A_1^i A_2^j \gamma_{\dot{a}\dot{b}}^{ij}) = \frac{9}{2}\left[(A_1 \cdot A_2)^2 - (A_1)^2 (A_2)^2\right]^2. \tag{31}$$

Finally, we can make use of the subset of the $SO(9,1)$ transformations,

$$\delta A_3^i = \omega_{ij} A_3^j + \omega_{i-} A_3^- + \omega_{i+} A_3^+ \tag{32}$$

that do not spoil either of the conditions, (25) or (29). The ω_{ij} transform of $\dot{\psi}_3^{\dot{a}}$, violates (29) and the ω_{i-} transform of A^i violates (25), but the ω_{i+} boosts can be used to impose the condition,

$$A_3^i = 0. \tag{33}$$

This replaces the variables A_3^i by ω^{i-}, introducing a jacobian,

$$\det \frac{\partial A_3^i}{\partial \omega^{j-}} = (A_3^+)^8. \tag{34}$$

When expressed in terms of the coordinates implied by the conditions (25), (29), (33) the action becomes, $S' = S_b + S_f$, where,

$$\begin{aligned} S_b &= \frac{1}{4\kappa}\Big\{-A_3^- A_3^+ \left((A_1)^2 + (A_2)^2\right) + (A_3^+)^2 \left((A_1^-)^2 + (A_2^-)^2\right) \\ &\quad (A_1)^2 (A_2)^2 - (A_1 \cdot A_2)^2\Big\} \end{aligned} \tag{35}$$

$$\begin{aligned} S_f &= \frac{1}{\kappa}\Big\{A_2^i \psi^{1\dot{a}} \gamma_{\dot{a}a}^i \psi_3^a + A_2^- \psi_1^a \psi_3^a \\ &\quad - A_1^i \dot{\psi}_2^{\dot{a}} \gamma_{\dot{a}a}^i \psi^{3a} - A_1^- \psi_2^a \psi_3^a + A_3^- \psi_1^a \psi_2^a + A_3^+ \dot{\psi}_1^{\dot{a}} \dot{\psi}_2^{\dot{a}}\Big\}. \end{aligned} \tag{36}$$

The partition function, $\mathcal{Z}^{(2)}$, is now given by (again dropping a κ independent constant)

$$\mathcal{Z}_2^{(2)} = \int (J^3)^8 d^8\psi_3^a d^8\omega^{i-} dA_3^+ dA_3^- \frac{(A_3^+)^{10}}{Pf(A_1^i A_2^j \gamma^{ij})} Z', \tag{37}$$

where $(J^3)^8$ soaks up the $\dot{\rho}$ integrations, and Z' is defined by the internal integrations,

$$Z' = \int d^8A_1^i d^8A_2^i d^8\dot{\psi}_1^{\dot{a}} d^8\dot{\psi}_2^{\dot{a}} d^8\psi_1^a d^8\psi_2^a \exp(-S'). \tag{38}$$

The fermionic fields can be integrated out by first redefining ψ_1 and ψ_2 by,

$$\psi_1^a \to \psi_1^a - \frac{A_1^-}{A_3^-}\psi_3^a, \qquad \psi_2^a \to \psi_2^a - \frac{A_2^-}{A_3^-}\psi_3^a \tag{39}$$

$$\dot\psi_1^{\dot a} \to \dot\psi_1^{\dot a} - \frac{A_1^i}{A_3^+}\gamma^i_{\dot a a}\psi_3^a, \qquad \dot\psi_2^{\dot a} \to \dot\psi_2^{\dot a} - \frac{A_2^i}{A_3^+}\gamma^i_{\dot a a}\psi_3^a. \tag{40}$$

This completes the square on ψ_3^a so that substituting (39), (40) into (36) gives,

$$S_f = \frac{1}{\kappa}\left\{A_3^-\psi_1^a\psi_2^a + A_3^+\dot\psi_1^{\dot a}\dot\psi_2^{\dot b} - \frac{A_1^iA_2^j}{A_3^+}\psi_3^a\gamma^{ij}_{ab}\psi_3^b\right\}. \tag{41}$$

Integration over ψ_1 and ψ_2 gives a factor of $\kappa^{-16}(A_3^-A_3^+)^8$ in the measure.

At this stage we could soak up the ψ_3^a's with a source term, which is precisely the situation we discussed earlier and the resulting bosonic integrations lead to (8) after a Wick rotation. However, we see that the integration over ψ_3^a is non-zero – in fact, from the last term in (41) the integration over these components gives a factor $(\kappa A_3^+)^{-4}Pf(A_1^iA_2^j\gamma^{ij})$, cancelling the Pfaffian in (37). This eliminates all the fermionic fields from the action, leaving the eight components of $\dot\rho^{\dot a}$ as the only fermionic integration variables. At this point the bosonic integrations bear a close similarity to the Minkowski signature version of the integrations carried out earlier, but with a different parameterization of the internal variables, A_1 and A_2. The integral over A_1^- and A_2^- is Gaussian and gives a factor of $\kappa(A_3^+)^{-2}$ so that the integral (37) reduces to

$$\mathcal{Z}_2^{(2)} = \int (\dot J_3)^8 d^8\omega^{i-}\, dA_3^+dA_3^-\, (A_3^+)^{12}(A_3^-)^8\Sigma, \tag{42}$$

where

$$\Sigma = \int d^8A_1^id^8A_2^i\, \exp(-S_b) \tag{43}$$

Before integration over the remaining internal coordinates A_1^i and A_2^i we want to make contact with the euclidean theory by replacing A_3^0 by iA_3^{10}, which has the effect of replacing $A_3^\pm$ by

$$A = \frac{1}{\sqrt2}(A_3^9 + iA_3^{10}), \qquad \bar A = \frac{1}{\sqrt2}(A_3^9 - iA_3^{10}). \tag{44}$$

With these conventions we make the replacement,

$$-A_3^-A_3^+ \to (A_3^9)^2 + (A_3^{10})^2 \equiv L^2 = A\bar A. \tag{45}$$

The ω^{i-} variables are now identified with the rotation generators that transform the commuting $SO(2)$ and $SO(8)$ subgroups of $SO(10)$ into each other.

The integration over A_1^i and A_2^i can now be carried out by transforming to eight-dimensional polar coordinates,

$$A_1^i = R_1 n_1^i, \qquad A_2^i = R_2 n_2^i, \tag{46}$$

where $n_1^2 = n_2^2 = 1$ and $n_1 \cdot n_2 = \cos\theta$.

$$\Sigma = \int d\Omega^7 d\Omega^6 \int dR_1 dR_2 d\theta R_1^7 R_2^7 (\sin\theta)^6 \ \exp(-S_b), \tag{47}$$

where the action is

$$S_b = \frac{1}{4\kappa}\left\{L^2(R_1^2+R_2^2) + (R_1)^2(R_2)^2\sin^2\theta\right\}. \tag{48}$$

The evaluation of the R_1, R_2 and θ integrals follows closely the steps from (14) to (17), leading to $\Sigma(L) = V(|L|)$. The two-instanton partition function (42) then reduces to

$$\mathcal{Z}_2^{(2)} = \kappa^{-11} \int dA d\bar{A}\ d^8 A_3^i\ A^{-4}\ e^{-V(|A_3|)}(J_3)^8, \tag{49}$$

which is of the form (9). Although this expression is not manifestly $SO(10)$ invariant it should give rise to covariant scattering amplitudes (since the external sources that soak up the eight fermionic zero modes also have a noncovariant form in the light-cone gauge).

This expression again has an obvious translation into the string theory D-instanton free energy, which is obtained by summing over closed world-sheets with boundaries on the two D-instantons. Contact is made with the light-cone parameterization by Fourier transforming with respect to $Y^- = Y_1^- - Y_2^-$ and transforming to the light-cone parameterization of [20] in which the surfaces are flat apart from the interaction points at which closed strings split and join. Each term in this series has the topology of a closed-string diagram, but with the initial and final closed-string states located at the finite times Y_1^+ and Y_2^+. The leading term in this sum – the cylinder diagram – is non-zero when eight undotted fermionic open-string states are attached to the boundaries. These are supplied by the $(J^3)^8$ term. Attaching these eight fermions is analogous to picking out the (velocity)^4 term in considering the force between D-particles in [16, 17, 18, 19]. The expression for this cylinder contribution is given in [8, 7]. Only the massless closed-string states contribute – just as in the D-particle case – and the result is proportional to $(p^+)^4 L^{-8}$, where $p^+ \sim \partial/\partial L$, and the result behaves as L^{-12}. The same result is obtained by viewing the process in the annulus channel, which is a trace over states of open strings with fixed end-points. Only the lowest states of the stretched strings contribute here also – precisely the same supermultiplet that enters into the Yang–Mills action. The leading diagram is the one-loop eight-point function with external ψ_3^a's which have vertices of the form $A_1^i \psi_2^{\dot{a}} \gamma_{\dot{a}a}^i \psi_3^a - A_2^i \psi_1^{\dot{a}} \gamma_{\dot{a}a}^i \psi_3^a$. The $A_{1,2}^i$ propagators are $1/L^2$ while the $\psi_{1,2}^{\dot{a}}$ propagators are $1/L$, so the diagram is proportional to $1/L^{12}$ in agreement with the cylinder interpretation.

The higher-order terms in the sum over the Yang–Mills diagrams translate into world-sheets which contain both planar and non-planar contributions – the latter encoding the effects of handles, or gravitational corrections. Although the light-cone treatment does not easily generalize to $N > 2$, the string world-sheet calculation has an obvious covariant formulation. It should therefore be possible to understand the counting of fermionic modes for general N in a more covariant manner – possibly by embedding the system in one with more local symmetry, such as the κ symmetry. The limit of large N, where closed-string loop corrections are suppressed if κN^2 is fixed, might be particularly interesting to analyze.

Finally, it is salutory to remember that the Yang–Mills approximation to the configuration space does not include the winding modes of Dirichlet open strings. These must play an important rôle in the compactification to nine dimensions on a circle of radius R. The Dirichlet open-string has winding numbers which transform under T-duality into momenta spaced by $\sqrt{\alpha'}/R$. The instanton then describes the euclidean compactification of the type IIA D-particle world-line. In the limit $R \to 0$, the one-dimensional Yang–Mills theory describing ten-dimensional D-particles is recovered. Clearly, the considerations of this paper must be generalized at scales $L \leq R$ to include the open-string winding modes, which are discrete momenta for the super Yang–Mills theory. This should lead to a cross-over between effects that arise at the ten-dimensional Planck scale in the type IIB theory and the eleven-dimensional Planck scale in type IIA, analogous to the cross-over described in [19].

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